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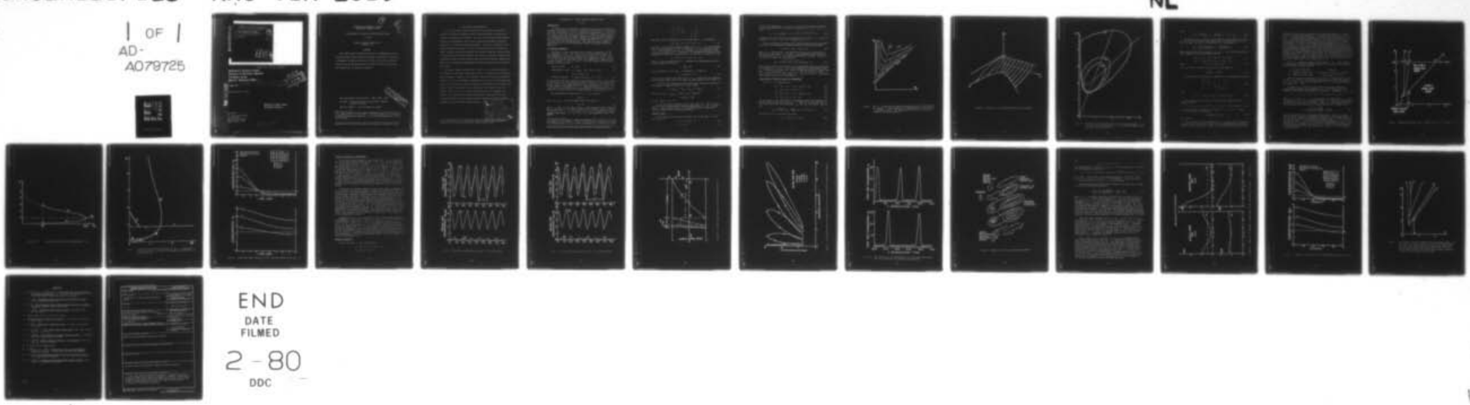

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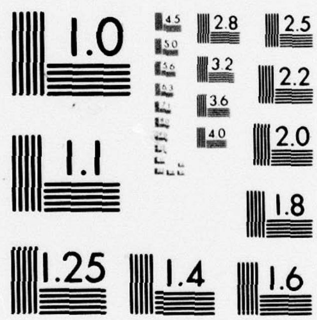

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10 R. Aris

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Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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BIFURCATIONS OF A MODEL DIFFUSION-REACTION SYSTEM

R. Aris*

Technical Summary Report #2019
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ABSTRACT

After some general observations on kinetic, compartmental and distributed systems a model suggested by Rössler is explored. Bifurcations to unsymmetric steady and uniform oscillatory states from the uniform steady states are demonstrated and the former are shown to break down into oscillations of increasing complexity.

AMS (MOS) Subject Classifications - 35K55, 58F14, 58F22

Key Words - Diffusion, Reaction, Bifurcation, Stability, Periodic Solutions

Work Unit Number 2 - Other Mathematical Methods

*This report, written during a summer appointment at MRC, is based on the Ph.D. dissertation of C. R. Kennedy. It will be presented at a conference on dynamical systems (Asilomar, Dec. 9-14, 1979) and published as a joint paper by SIAM.

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SIGNIFICANCE AND EXPLANATION

The 70's have seen a surge of interest in the behavior of solutions of systems of differential equations which in some sense model the physics and chemistry of diffusion and reaction. One line of work has grown from Turing's approach to morphogenesis, which he saw as arising from stable non-uniform solutions in an otherwise uniform environment. Another was stimulated by the Belousov-Zhabotinski reaction, a colorful oscillating reaction of moderately exotic chemicals, and by the discovery that catalytic reactions were more prone to oscillation than had been realized. Both lines are obviously connected with bifurcation theory (the latter particularly with that of Poincaré, Andronov and Hopf), but the developing understanding of strange attractors has also begun to play a part.

The present example, basically due to Rössler, is an attempt to exhibit this behavior in as simple a system as possible. Two equations with three parameters represent a system of two "reactions", only one of which has a mild non-linearity. One of the parameters corresponds to the ratio of the characteristic times in the two equations and has no effect on the position of the steady state but does control its stability. The character of the solution in one cell, several coupled cells or a continuum under variation of this parameter is studied; it is found that the uniform steady state breaks down either into a uniform oscillation or non-uniform steady state and that the latter further bifurcates into increasingly complex behavior.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

BIFURCATIONS OF A MODEL DIFFUSION-REACTION SYSTEM

R. Aris*

INTRODUCTION.

The bifurcation of solutions to equations representing the interplay of diffusion and reaction has received a great deal of attention in recent years both from a theoretical (1, 2, 3, 4) and from an experimental (5, 6) point of view, while, since the discovery of chaos (7, 8) many have wondered whether the strange flickerings and turbulent behavior of quite ordinary systems may not be ascribed to some such cause. Rössler has demonstrated how several types of system might be patient of strange attractors (9, 10) and we wish to pursue one of his suggestions here. Marek has considered a somewhat similar system (11) while a bimolecular reaction has engaged Othmer and Kádas (12).

THE SYSTEMS CONSIDERED.

The vector $\underset{\sim}{x}$ may be thought of as a vector of compositions in $\mathbb{R}^n$ with components $x^1, x^2, \ldots, x^n$. It may exist in three contexts which are worth comparing: the kinetic, where it is a function only of time, $\underset{\sim}{x}(t)$; the compartmental, where it also depends on an enumerative index, $\underset{\sim}{x}_j(t)$, $j = 1,\ldots,N$; and the distributed, where (for simplicity) it depends on one position variable, z say, as $\underset{\sim}{x}(z,t)$. The three habitats will be distinguished by their equations:

Kinetic system: $$\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x}) \tag{K}$$

Compartmental system: $$\dot{\underset{\sim}{x}}_j = \underset{\sim}{D}(\underset{\sim}{x}_{j+1} - 2\underset{\sim}{x}_j + \underset{\sim}{x}_{j-1}) + \underset{\sim}{f}(\underset{\sim}{x}_j) \tag{C}$$

$$\underset{\sim}{x}_0 = \underset{\sim}{x}_1, \quad \underset{\sim}{x}_{N+1} = \underset{\sim}{x}_N$$

Distributed system: $$\dot{\underset{\sim}{x}} = \underset{\sim}{D}\underset{\sim}{x}_{zz} + \underset{\sim}{f}(\underset{\sim}{x}), \quad \underset{\sim}{x}_z = 0, \; z = 0, L . \tag{D}$$

We observe that all three systems are closed except in so far as the function $\underset{\sim}{f}$ may contain terms which imply the removal or supply of any species. The kinetic system is the single compartment; the distributed is the limit of an infinite number of compartments if $\underset{\sim}{D}$ is made proportional to (N/L). Of course there are more subtle ways of approximating the distributed system by a discrete one and we shall use a Galerkin method in calculations on D.

The equilibrium or steady state of K is $\underset{\sim}{x}_e$ given by

$$\underset{\sim}{f}(\underset{\sim}{x}_e) = 0 \tag{1}$$

and if $\underset{\sim}{x} = \underset{\sim}{x}_e + \underset{\sim}{\varphi}$ the linearization about this state is

$$\dot{\underset{\sim}{\varphi}} = \underset{\sim}{J}\underset{\sim}{\varphi} \tag{2}$$

where $\underset{\sim}{J}$ is the Jacobian matrix $\nabla\underset{\sim}{f}(\underset{\sim}{x}_e)$. Local stability is then determined by the eigenvalues of $\underset{\sim}{J}$. These solutions are also steady states of (C) and (D) with $\underset{\sim}{x}_j = \underset{\sim}{x}_e$, $j = 1,\ldots,N$, $\underset{\sim}{x}(z,t) = \underset{\sim}{x}_e$, $0 \leq z \leq L$, and will be called <u>uniform</u> steady states. If κ_j, $j = 1,2,\ldots,N$ are eigenvalues of the $N \times N$ matrix

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$$\begin{bmatrix} 1 & -1 & . & . & . & . \\ -1 & 2 & -1 & & . & . \\ . & -1 & 2 & -1 & . & . \\ . & . & -1 & . & . & . \\ . & . & . & . & 2 & -1 \\ . & . & . & . & -1 & 1 \end{bmatrix}$$

then those of the linearization of the uniform state of C are given by

$$|\lambda \underset{\sim}{I} + \kappa_j \underset{\sim}{D} - \underset{\sim}{J}| = 0, \; j = 1,2,\ldots,N \tag{3}$$

Since the κ_j are distinct positive numbers with $\kappa_1 = 0$, the first equation gives the eigenvalues of (K) and the remaining $(N - 1)n$ eigenvalues depend on $\underset{\sim}{D}$. Thus the stability of (K) is a necessary, but not sufficient, condition for the stability of (C). Moreover it is differences of diffusivity that destabilize (C) when (K) is stable, for if $\underset{\sim}{D} = D\underset{\sim}{I}$ the nN roots of (3) are

$$\lambda_i - \kappa_j D, \; i = 1,\ldots,n, \; j = 1,\ldots,N,$$

where λ_i are the eigenvalues of $\underset{\sim}{J}$. Similarly the linearization about the uniform steady state of (D) is

$$\dot{\underset{\sim}{\varphi}} = \underset{\sim}{D}\underset{\sim}{\varphi}_{zz} + \underset{\sim}{J}\underset{\sim}{\varphi} \tag{4}$$

and a perturbation of the form $\underset{\sim}{c}e^{\lambda t}\cos(n\pi z/L)$ is stable if

$$|\lambda \underset{\sim}{I} + (n\pi/L)^2 \underset{\sim}{D} - \underset{\sim}{J}| = 0 \tag{5}$$

has roots with negative real parts for all n. It follows that the stability of the uniform steady state of (D) bears the same relation to (K) as does (C).

(C) and (D) also have non-uniform steady states unrelated to those of (K). They appear in symmetric pairs, for if x_{ej} satisfy

$$0 = \underset{\sim}{D}(\underset{\sim}{x}_{e,j+1} - 2\underset{\sim}{x}_{e,j} + \underset{\sim}{x}_{e,j-1}) + \underset{\sim}{f}(\underset{\sim}{x}_{e,j}) \tag{6}$$

then so do $\underset{\sim}{x}^*_{e,j} = \underset{\sim}{x}_{e,N+1-j}$, and if $\underset{\sim}{x}_e(z)$ satisfies

$$0 = \underset{\sim}{D}(\underset{\sim}{x}_e)_{zz} + \underset{\sim}{f}(\underset{\sim}{x}_e) \tag{7}$$

so does $\underset{\sim}{x}^*_e(z) = \underset{\sim}{x}_e(L - z)$.

If (K) has periodic solutions $\underset{\sim}{x}(t) = \underset{\sim}{p}(t) = \underset{\sim}{p}(t + T)$, then (C) and (D) will also have uniform periodic solutions independent of j and z respectively. Similar remarks apply to the linearized stability which must of course be determined by the Floquet multipliers.

RÖSSLER'S MODEL

For the model system proposed by Rössler (10) we will take $\underset{\sim}{x} = (u,v)$ and then (K) is

$$\dot{u} = u + 1 - uv/(\kappa + u) \tag{8}$$

$$\dot{v} = \nu(\sigma u - v) \tag{9}$$

and has three parameters κ, ν, and σ. On the first and last of these the steady state depends and is unique and "physical" (i.e. $u_e > 0$, $v_e > 0$) when $\sigma > 1$

$$v_e = \sigma u_e = \frac{\sigma}{2(\sigma-1)}\{(1+\kappa) + \sqrt{(1+\kappa)^2 + 4\kappa(\sigma-1)}\} \tag{10}$$

It can be most easily found as the intersection of the line $v = \sigma u$ with the hyperbola $v = u + (1+\kappa) + \kappa u^{-1}$ as shown in Fig. 1.

Of the conditions for stability one (that the product of the eigenvalues be positive) is automatically satisfied and the other is satisfied when

$$\nu > g(\kappa,\sigma) = (u_e^2 - \kappa)/u_e(\kappa + u_e) \tag{11}$$

where u_e is given by (10). The dotted lines in Fig. 1 are contours of ν and in Fig. 2 the surface g is shown. If the parametric point lies below the surface shown in Fig. 2 the unique steady state is unstable. Moreover as we cross this surface the eigenvalues are

$$\pm i\omega = \pm i\sqrt{(u_e^2 - \kappa)(2\kappa + u_e + \kappa u_e)/u_e(u_e + \kappa)} \tag{12}$$

Applying the criterion for stability of the bifurcating limit cycle shows that it is stable for $\nu < g(\kappa,\sigma)$ and in the neighborhood of that surface. Thus as ν decreases the steady state loses its stability and sheds a stable limit cycle. Fig. 3 shows this for $\sigma = 2.5$, $\kappa = 0.144$ for which $u_e = 0.87$, $v_e = 2.18$ and $g = 0.7$; limit cycles for $\nu = 0.6$, 0.5 and 0.3 are shown. These become larger and more triangular as $\nu \to 0$.

STEADY STATES OF RÖSSLER'S MODEL IN COMPARTMENTS.

For $N = 2$ the equations are:

$$\dot{u}_1 = \tilde{\omega}(u_2 - u_1) + 1 + u_1 - u_1 v_1/(\kappa + u_1) \tag{13}$$

$$\dot{v}_1 = \mu(v_2 - v_1) + \nu(\sigma u_1 - v_1) \tag{14}$$

$$\dot{u}_2 = \tilde{\omega}(u_1 - u_2) + 1 + u_2 - u_2 v_2/(\kappa + u_2) \tag{15}$$

$$\dot{v}_2 = \mu(v_1 - v_2) + \nu(\sigma u_2 - v_2) \tag{16}$$

For the moment we will follow Rössler in considering only the exchange of the second chemical species and so put $\tilde{\omega} = 0$. Since (14) and (16) are linear they can be solved for the steady-state v_{1e} and v_{2e} in terms of u_{1e} and u_{2e}

$$v_{je} = \frac{\sigma(\mu+\nu)}{\nu + 2\mu} u_{je} + \frac{\sigma\mu}{\nu + 2\mu} u_{ie}, \quad i \neq j; i,j = 1,2 \tag{17}$$

Substituting into (13) and (15) then gives

$$u_{2e} = F(u_{1e}), u_{1e} = F(u_{2e}) \tag{18}$$

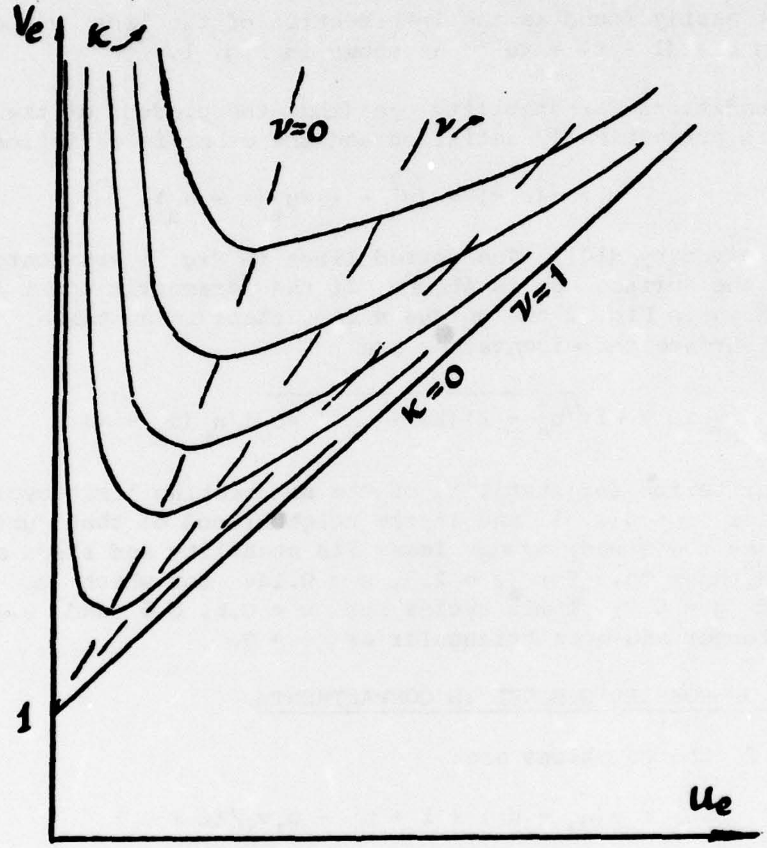


Figure 1 The u, v-plane for the system (K) showing the loci of steady-states for various κ. These curves must be intersected by a line of slope σ through the origin. The broken lines show the loci of critical stability for various values of ν.

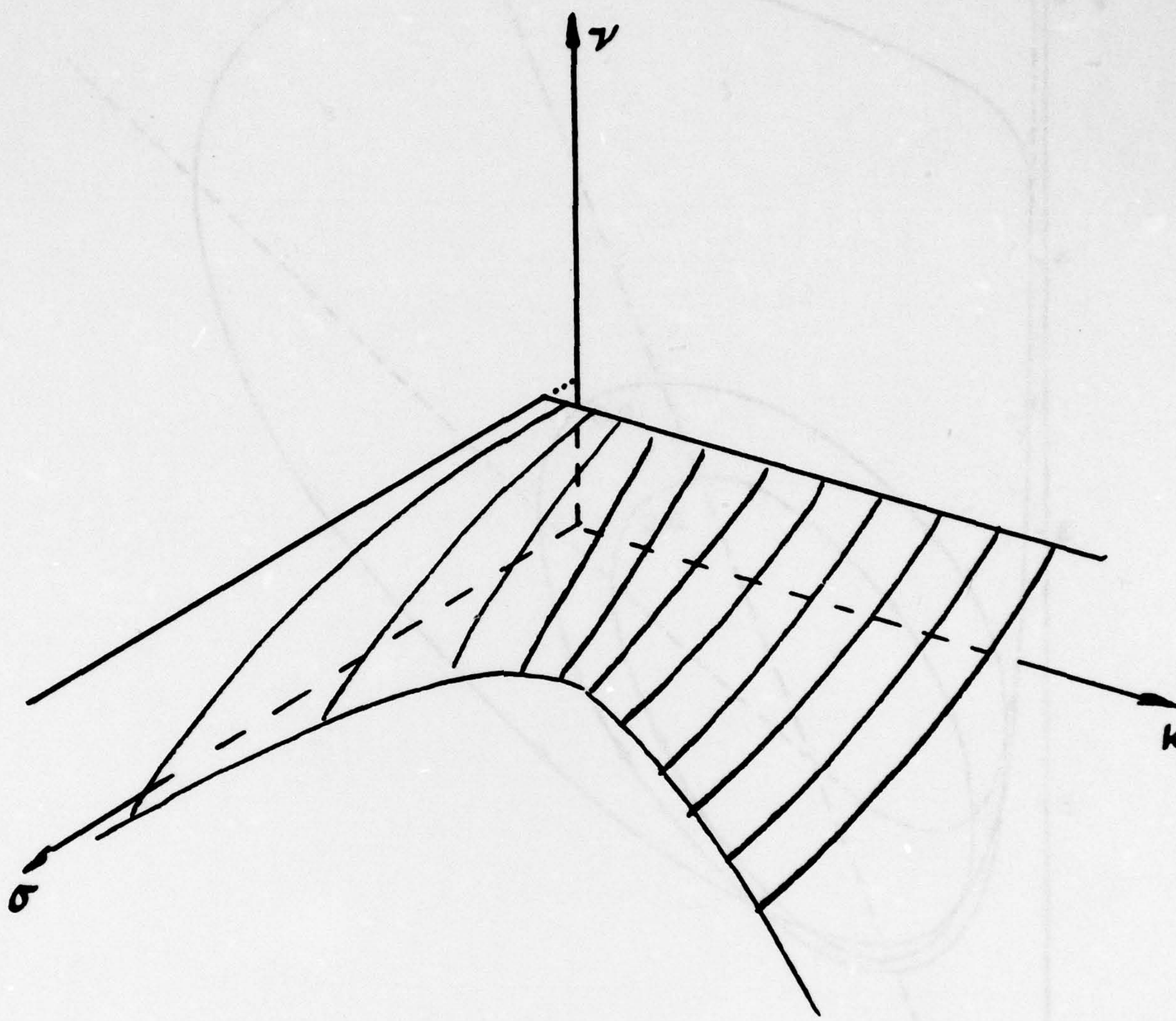


Figure 2 Surface in κ, ν, σ-space below which (K) is unstable.

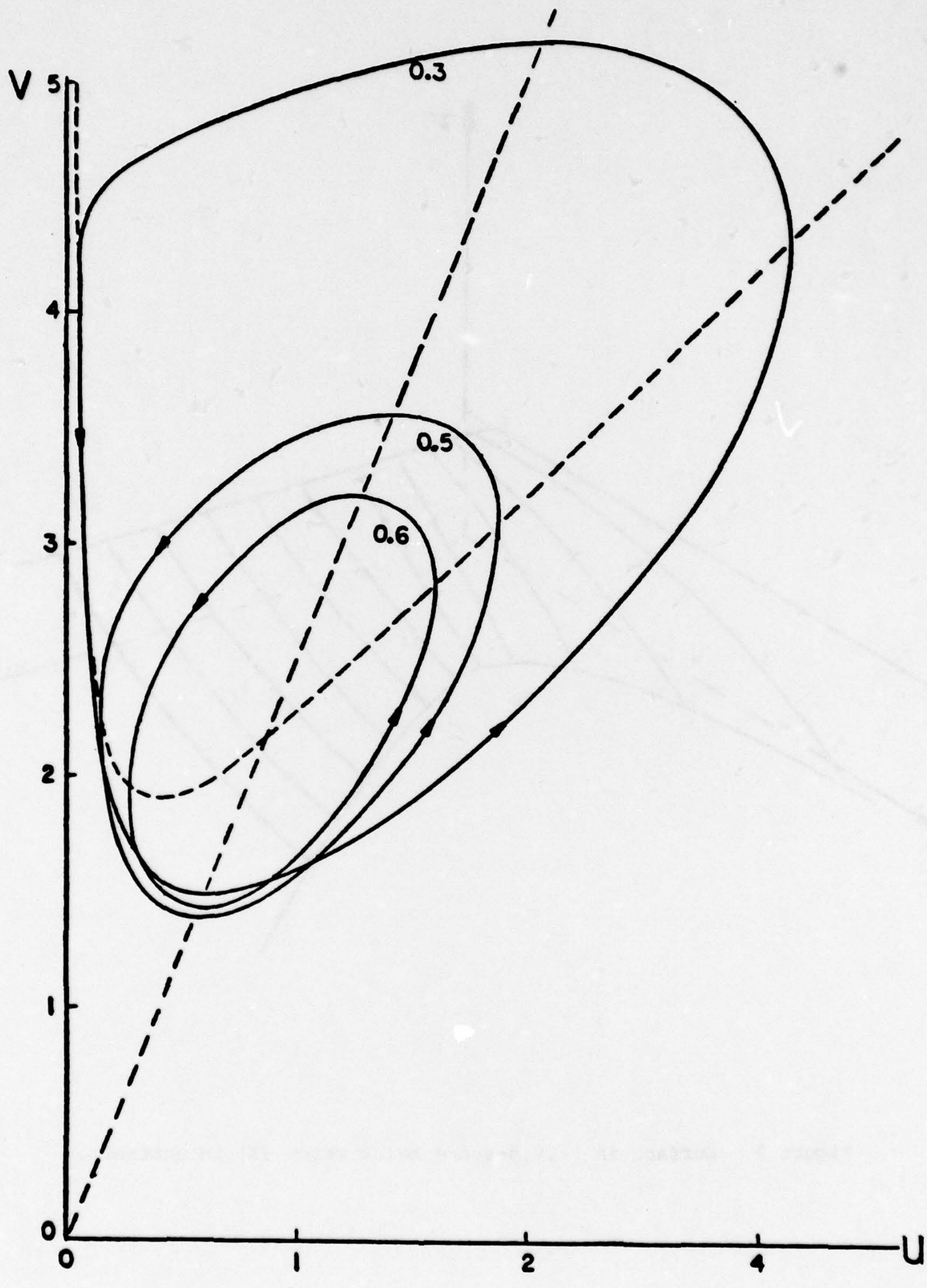


Figure 3 Steady-state and three limit cycles for the system (K) with $\sigma = 2.5$ $\kappa = 0.144$. The critical value of ν is 0.7 and the numbers on the limit cycles are the values of ν.

where

$$F(u) = \frac{\alpha + 2}{\sigma}\left\{\frac{\kappa}{u} + \left[1 - \sigma\frac{\alpha + 1}{\alpha + 2}\right]u + 1 + \kappa\right\}, \ \alpha = \frac{\nu}{\mu}. \tag{19}$$

The equation $u_{je} = F(F(u_{je}))$ breaks down into two quadratics, the first of which is the one already encountered giving the uniform state $u_{je} = u_e$, $v_{je} = v_e$ where u_e and v_e are given by (10). The second is

$$u_e^2 + \frac{(1 + \kappa)(\alpha + 2)}{(\alpha + 2) - \sigma(\alpha + 1)} u_e - \frac{\kappa(\alpha + 2)}{\sigma\alpha - (\alpha + 2)} = 0 \tag{20}$$

When the two roots of this quadratic (u_+ and u_-, say) are both positive we have a symmetric pair of non-uniform solutions

$$\begin{aligned} u_{1e} &= u_+, \ v_{1e} = f(u_+), \ u_{2e} = u_-, \ v_{2e} = f(u_-) \\ u_{1e} &= u_-, \ v_{1e} = f(u_-), \ u_{2e} = u_+, \ v_{2e} = f(u_+) \end{aligned} \tag{21}$$

where $$f(u) = (1 + u)(\kappa + u)/u \tag{22}$$

It is not hard to see that such solutions can only obtain if

$$\frac{\alpha + 2}{\alpha + 1} < \sigma < \frac{\alpha + 2}{\alpha}. \tag{23}$$

The stability of the solutions is governed by the eigenvalue of the matrix

$$\begin{bmatrix} g_1 & -h_1 & \cdot & \cdot \\ \nu\sigma & -(\nu + \mu) & \cdot & \mu \\ \cdot & \cdot & g_2 & -h_2 \\ \cdot & \mu & \nu\sigma & -(\nu + \mu) \end{bmatrix} \tag{24}$$

where

$$g_j = 1 - \sigma v_{je}/(\kappa + u_{je})^2, \ h_j = u_{je}/(\kappa + u_{je}). \tag{25}$$

For the uniform steady state, the characteristic equation has two quadratic factors:

$$\{\lambda^2 + (\nu - g)\lambda + \nu(\sigma h - g)\}\{\lambda^2 + (\nu - g + 2\mu) \ + \nu(\sigma h - g) - 2\mu g\} = 0. \tag{26}$$

The first of these is the same as for (K) and $\sigma h - g > 0$, $\sigma > 1$. Since $\mu > 0$ we now have the conditions

$$\nu > \mathrm{Max}\{g, 2\mu g/(\sigma h - g)\} \tag{27}$$

for stability.

Again we expect interesting behavior as ν decreases and the bifurcation picture will be governed by the order in which the coefficients in the two quadratic factors change sign. There are two possibilities which can best be

illustrated by Fig. 4 which is drawn for the same fixed values of κ and σ as Fig. 3. The exchange parameter μ is also held constant so that decreasing ν is represented by a parametric point travelling horizontally to the left. The uniform steady state is stable to the right of ABC and if the parametric point cross AB the uniform state loses its stability to the uniform periodic state in which the two compartments behave exactly as in (K). The line (O)BC is $\nu = 2\mu g/(\sigma h - g)$ so that, if the parametric point cross BC, the last coefficient in (26) changes sign first and the system loses stability by one real eigenvalue becoming positive (rather than a pair of complex conjugates passing across the imaginary axis, as in the crossing of AB) and the bifurcating solution is a stable non-uniform steady state. In the region DEBC the non-uniform steady states are stable and the same state obtains over any ray through the origin, since u_+ and u_- depend only on ν/μ, as shown in Fig. 5. They lose their stability across the curves DE and EB as will be discussed after a look at the steady states of the system with $N > 2$.

For $N = 3$, a quadratic equation can be found for u_{2e} (at the expense of considerable algebra) and further equations for u_{1e}, u_{3e}. They give rise to three classes of steady solution:

I. uniform steady state (u_e, u_e, u_e)

II. asymmetric steady states (u_{1e}, u_{2e}, u_{3e}), (u_{3e}, u_{2e}, u_{1e}), $u_{1e} \neq u_{3e}$

III. symmetric steady states (u_{1e}, u_{2e}, u_{3e}), $u_{1e} = u_{3e}$

We shall not give the formulae but show in Fig. 6 the solutions that make all the u_{je} positive. For this set of parameters only Class I can obtain for $\nu/\mu > 1.45$; I and III are allowed for $.57 < \nu/\mu < 1.45$ and all three are possible for $\nu < .57\mu$.

For large N we can simplify the picture by being more general and allowing the exchange of both components i.e. $\tilde{\omega} \neq 0$. Then, by (3), the stability of the uniform steady state is governed by N quadratics, $j = 1, \ldots, N$,

$$\lambda^2 + \{\nu - g + \kappa_j(\tilde{\omega} + \mu)\}\lambda + \nu(\sigma h - g) + \kappa_j(\tilde{\omega}\mu\kappa_j + \tilde{\omega}\nu - \mu g) = 0$$

Since $\kappa_1 = 0$ and the κ_j, $j > 1$, are positive the coefficient of λ changes sign first for $j = 1$ as ν becomes less than g. If all the constant terms are positive, as will be the case when $\tilde{\omega}$ and μ are sufficiently small, the uniform steady state gives rise to a uniform periodic one. Across the line

$$\mu = \nu \operatorname*{Min^+}_{j>1} \left\{ \frac{\sigma h - g + \tilde{\omega}\kappa_j^2}{\kappa_j(g - \tilde{\omega}\kappa_j)} \right\} = \nu \operatorname*{Min^+}_{j>1} M_j \qquad (28)$$

there is a bifurcation to a non-uniform steady state. (Min^+ is the least positive value.) Since we are primarily interested in the dynamic behavior we will not try to explore this complex picture, but choose $\tilde{\omega}$ so as to simplify it. We note that $j = 2$ may not be the minimizing value of j in (28). For example with $N = 10$ and $\tilde{\omega} = 2$ only one of the expressions in (28) is positive and we have a diagram like that in Fig. 4. The non-uniform steady states in the DEBC region are the same along rays of constant ν/μ. Three examples are given in Fig. 7.

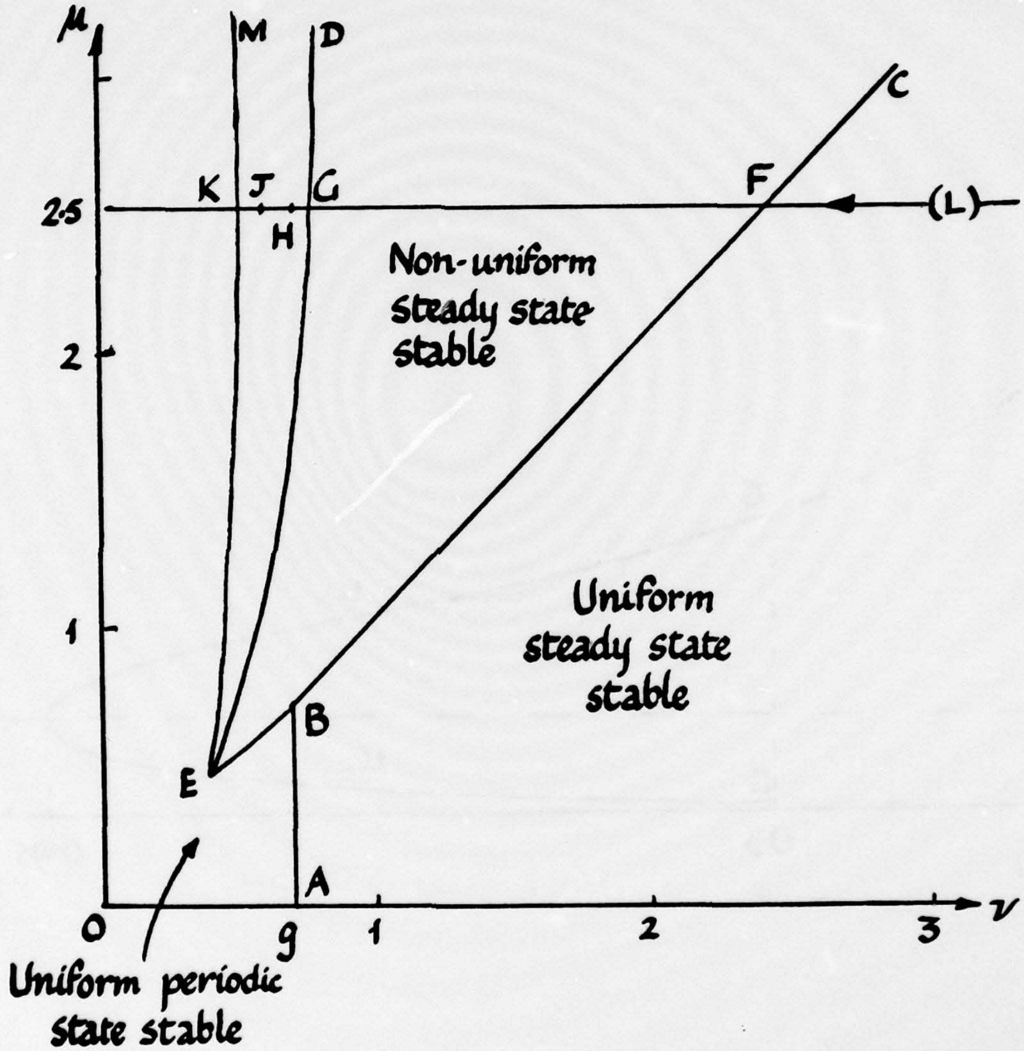


Figure 4 Bifurcation diagram in the μ,ν-plane for $N = 2$, $\kappa = 0.144$, $\sigma = 2.5$.

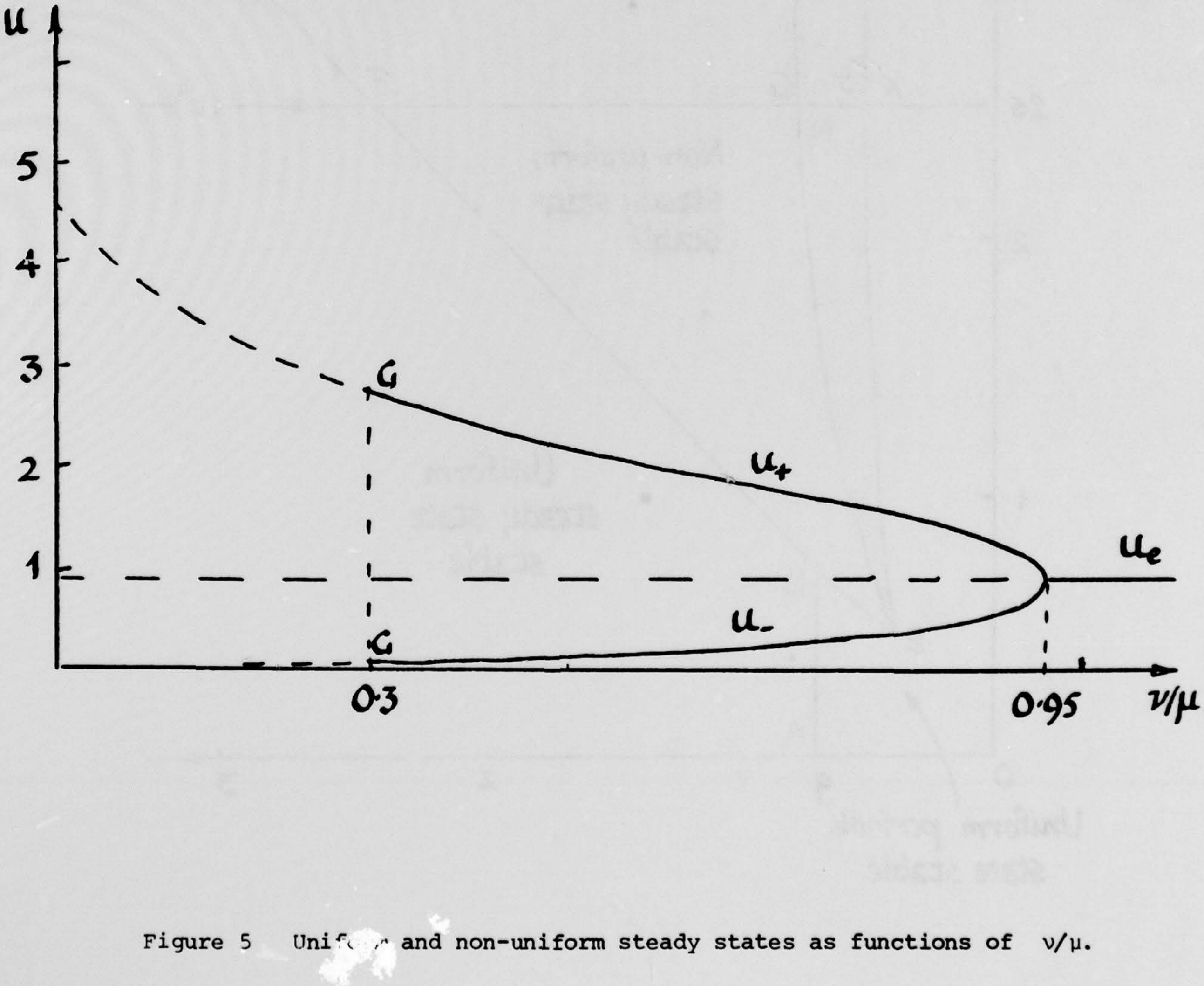


Figure 5 Unif[illegible] and non-uniform steady states as functions of ν/μ.

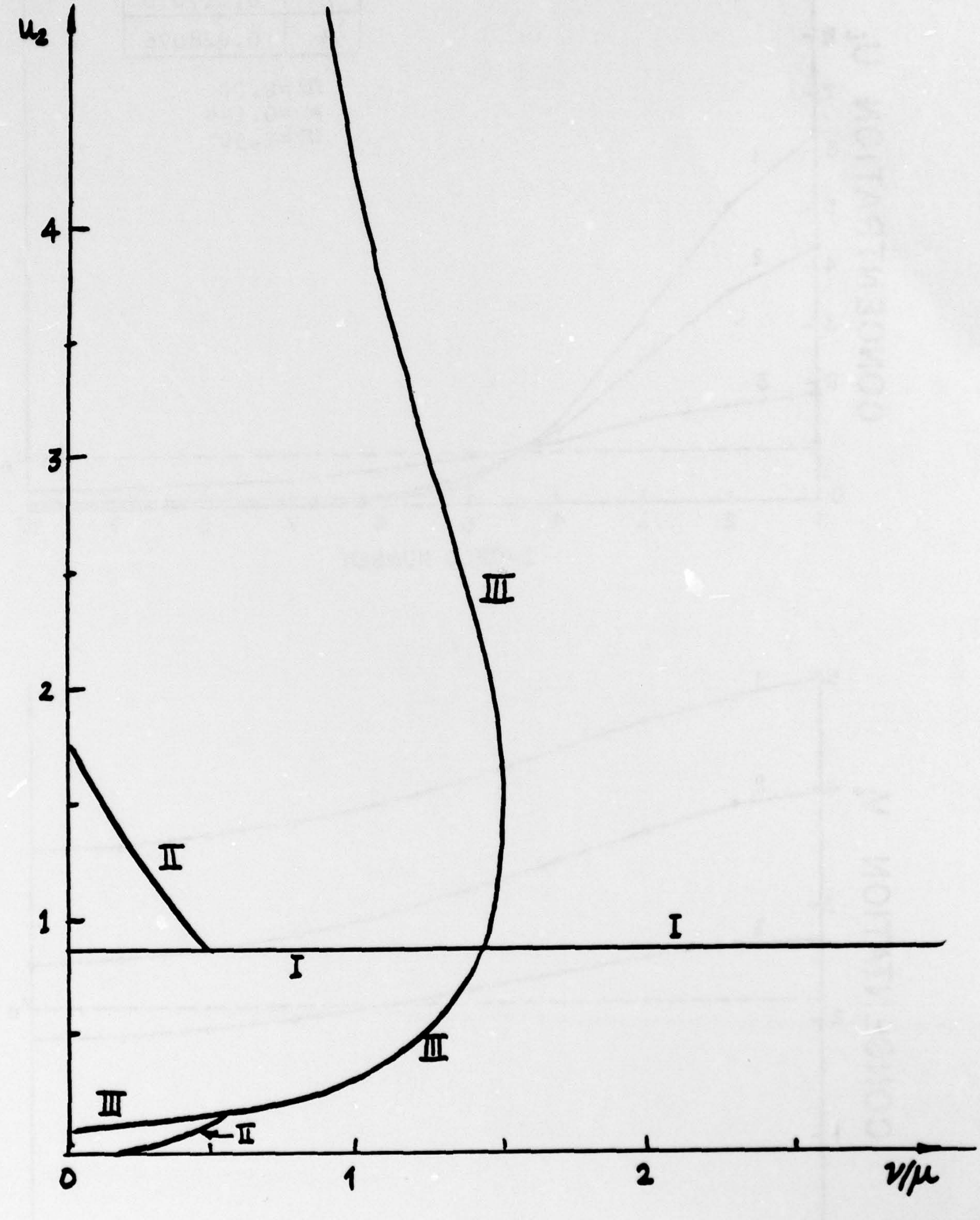


Figure 6 Uniform and non-uniform steady states for N = 3. I denotes the uniform steady state, II denotes the axisymmetric steady state, III the symmetric one.

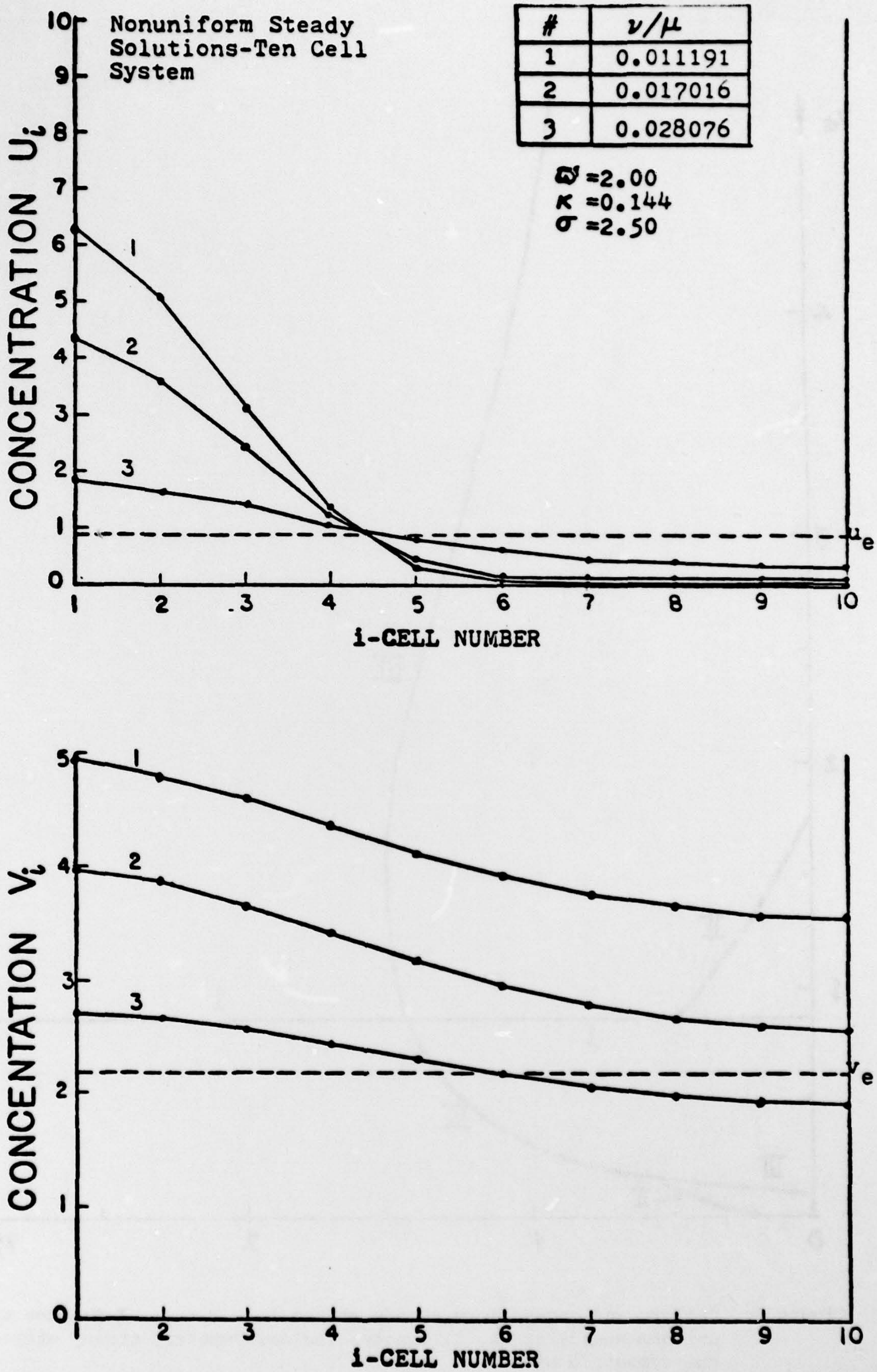


Figure 7 Non-uniform steady states for N = 10 and three values of the ratio ν/μ.

DYNAMICS OF RÖSSLER IN COMPARTMENTS.

Rössler gave several examples of the dynamics with $N = 2$ and we shall follow his lead by systematically varying ν for $N = 2$ before increasing N. In particular we keep $\kappa = 0.144$, $\sigma = \mu = 2.5$ and $\tilde{\omega} = 0$, so that we move to the left on the line (L) of Fig. 4. To the right of F there is only the uniform state $u_1 = u_2 = 0.87$, $v_1 = v_2 = 2.18$. Between F and G the non-uniform steady states can be followed in Fig. 5, but at G this becomes unstable by a Hopf bifurcation, ν being 0.755. The kind of oscillation that arises is shown in Fig. 8 for which $\nu = .75$ (point H in Fig. 4). At $\nu = .696$ (point J of Fig. 4), this oscillation loses its stability as a Floquet multiplier passes out of the unit circle through -1, and a stable periodic solution of twice the period arises; this solution, for $\nu = 0.690$, is shown in Fig. 9. By tracking the critical Floquet multiplier as in Fig. 10, we find a bifurcation which again doubles the period to about four times the original one. The stability of this quickly breaks down and the period again doubles.

At this point it becomes difficult to keep track of what is happening for, even when there is an attracting orbit, it takes an unconscionably long time to lock onto it. We tried recording the values of everything each time u_1 passed through a maximum and this poor-man's-Poincaré-map gives clear support for the picture just drawn. There was less clear evidence of period 3 oscillations but much indication of irregular behavior, though whether this is chaos or a solution of long period cannot be determined empirically. It may be that as ν decreases we have a sequence of bifurcations such as is known (12) to obtain for $x_{n+1} = \lambda x_n(1 - x_n)$ or there may be a Hopf bifurcation of the Poincaré map to a torus and so on. It is also possible that the successive bifurcations cease altogether for, when $\nu < 0.515$ (point K in Fig. 4), the uniform periodic solutions are stable. There would thus appear to be a locus EKM in Fig. 4 such that the uniform periodic solution is stable in the region OABEKM. The structure of left hand part of the zone MKEGD is as complex as it is mathematically erogenous.

Explorations for $N = 10$ give evidence that the same general behavior is to be expected. In the case $\kappa = 0.144$, $\mu = 250$, $\tilde{\omega} = 2$, $\sigma = 2.5$ the stable non-uniform limit cycles in five cells of the symmetric system are shown in Fig. 11 for $\nu = 0.6$. By $\nu = 0.57$ the solution has already bifurcated twice. For yet smaller values of ν the solution is shown in Fig. 12, where the smaller ripples (only u_1 is shown) are 'near' the uniform periodic solution and the large departures 'near' the non-uniform. Since the uniform periodic solution is ultimately stable the scheme of things would seem to be as sketched in Fig. 13, where as ν decreases the increasing stability and attractiveness of the uniform orbits tears the non-uniform orbit apart. This is in keeping with Rössler's observations (10).

RÖSSLER DISTRIBUTED.

Here we will take $\xi = z/L$ and work with

$$u_\tau = \tilde{\omega} u_{\xi\xi} + 1 + u - uv/(\kappa + u) \tag{29}$$

$$v_\tau = \mu v_{\xi\xi} + \nu(\sigma u - v) \tag{30}$$

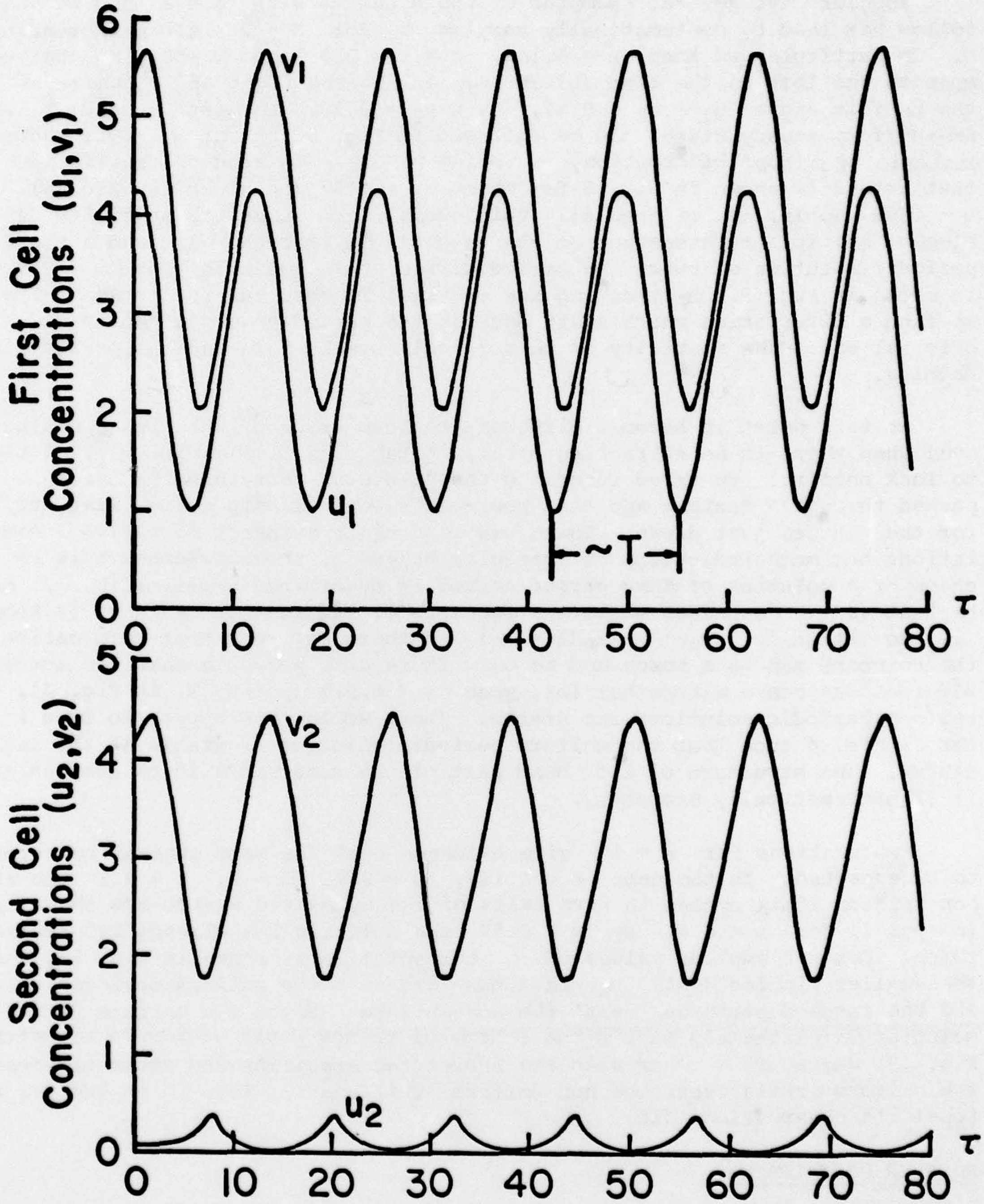


Figure 8 Non-uniform oscillations of period ~T in the two cells.

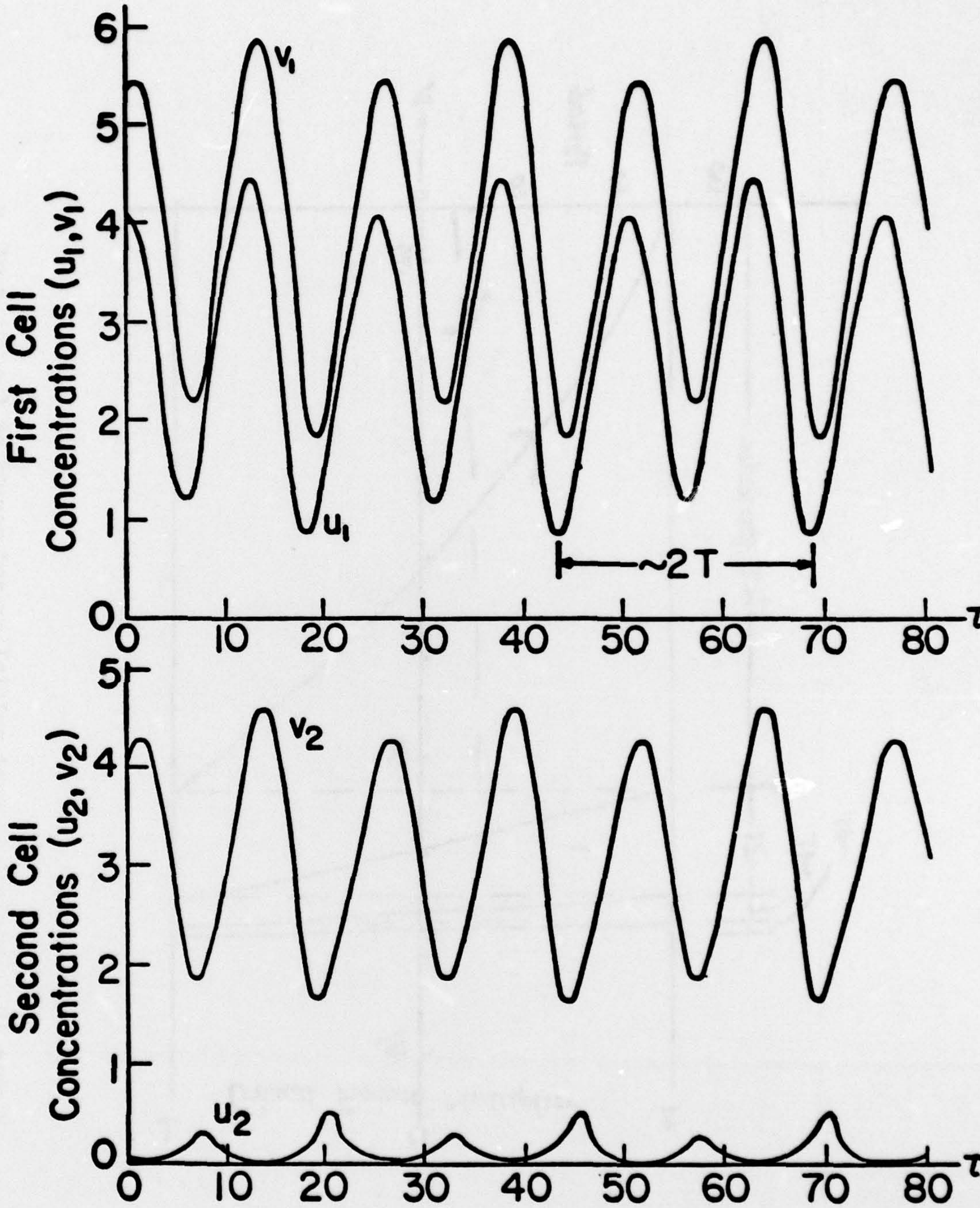


Figure 9 Non-uniform oscillations of period ~2T in the two cells.

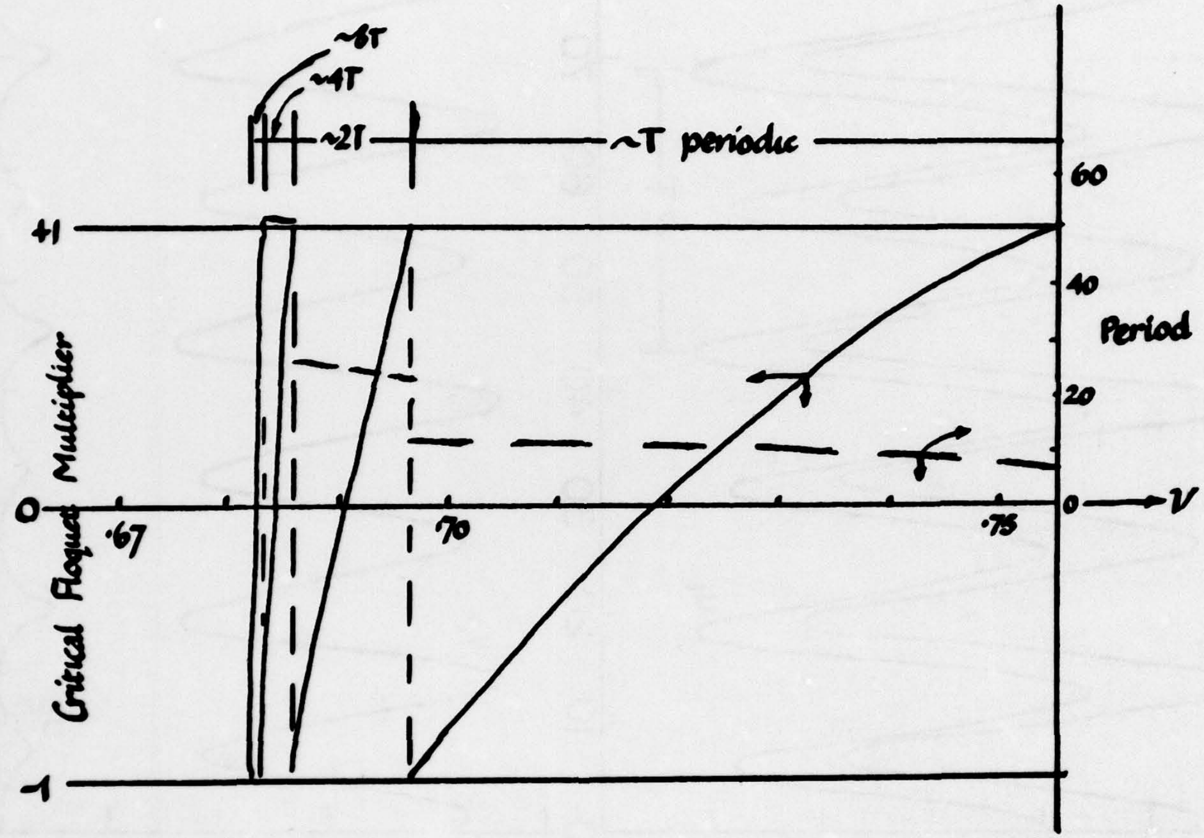


Figure 10 Critical Floquet multiplier and period as functions of ν.

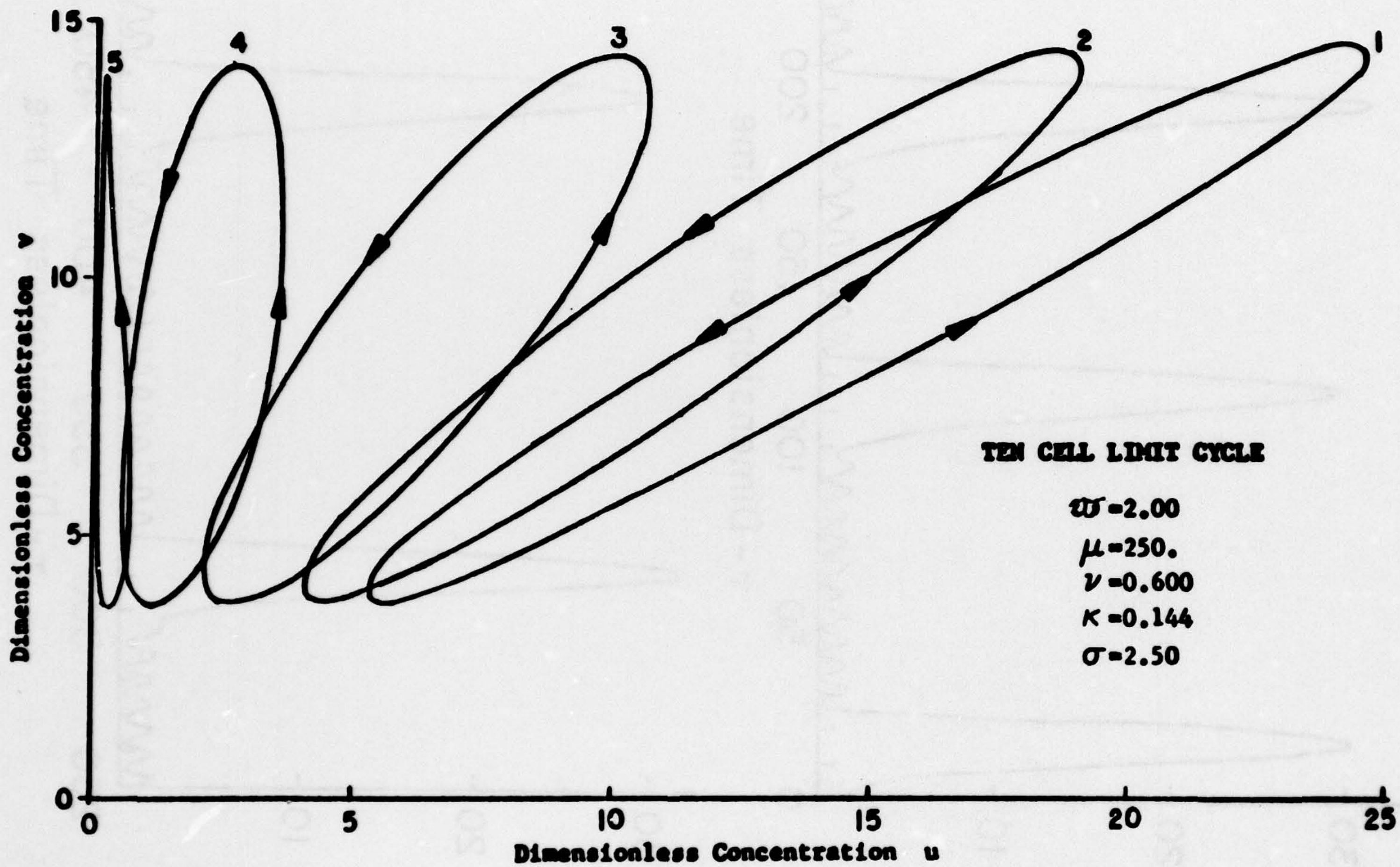
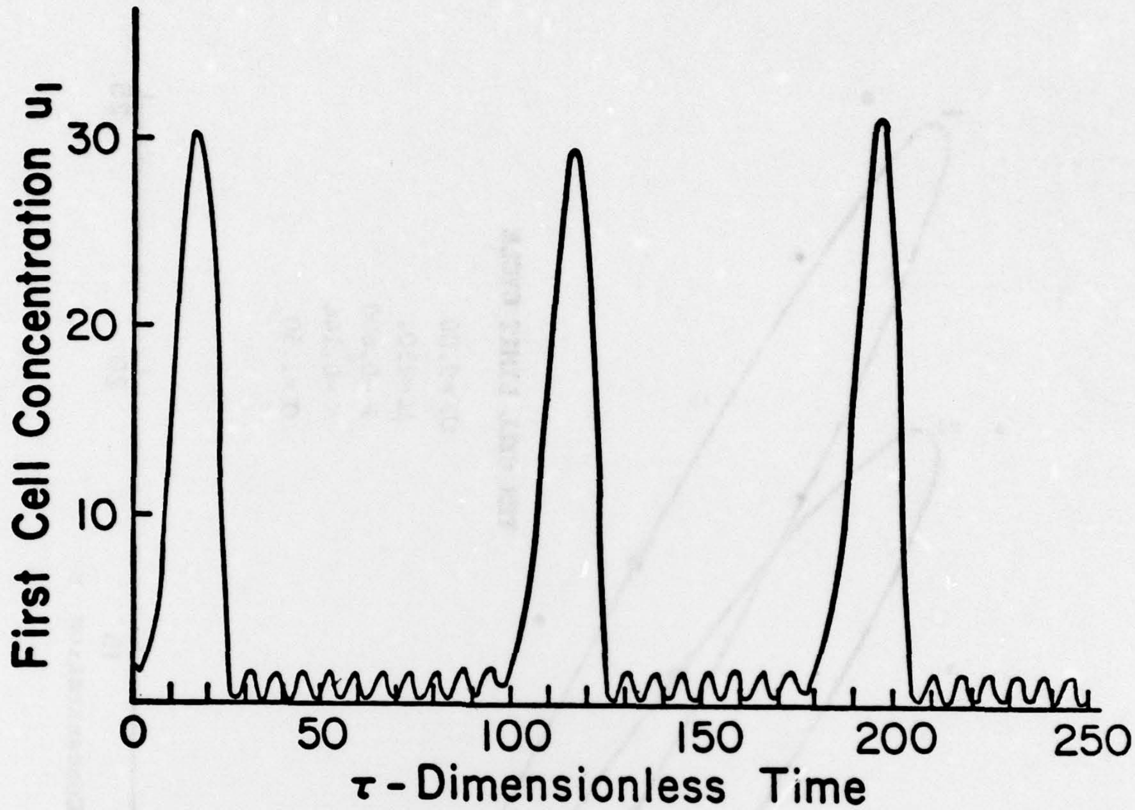


Figure 11 Non-uniform periodic solutions in five of the ten cells shown as phase-plane diagrams.



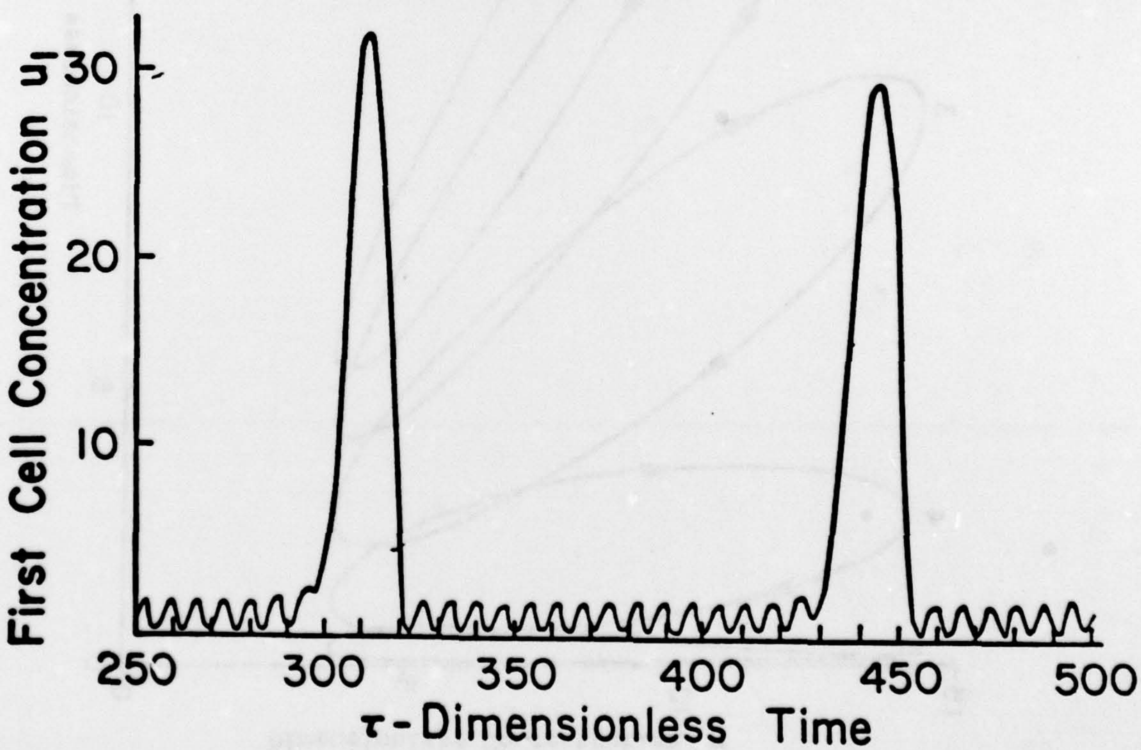


Figure 12 Time variation of the concentration u_1 for an orbit which spends most of its time near the uniform periodic solution.

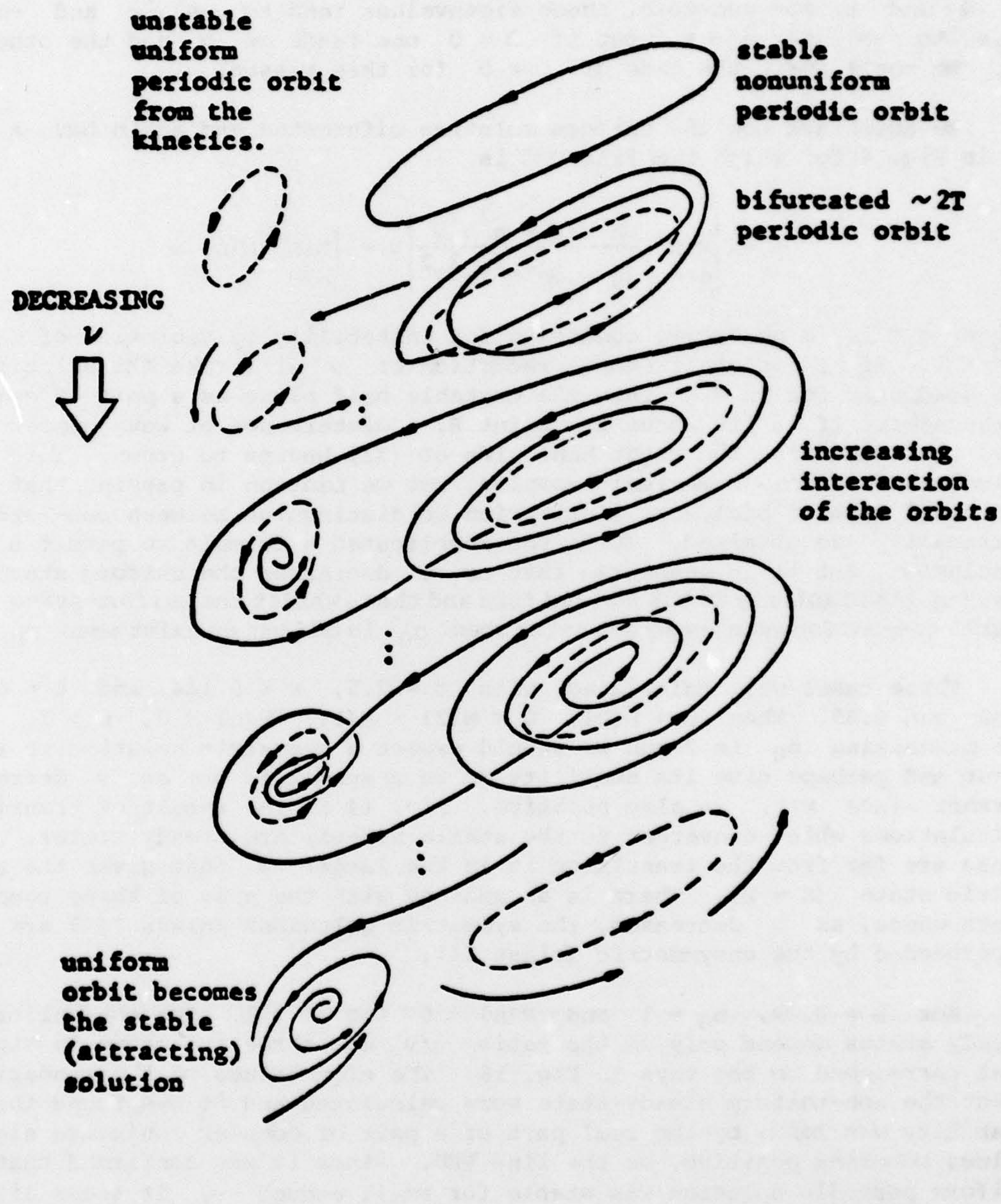


Figure 13 Schematic of how the attracting orbits evolve.

with

$$u_\xi = v_\xi = 0 \quad \text{at} \quad \xi = 0,1 \tag{31}$$

The uniform solution $u_e(\xi) = u_e$, $v_e(\xi) = v_e$ is still given by (10), while by (5), its stability is governed by

$$\lambda^2 + (\nu - g + (\tilde{\omega} + \mu)n^2\pi^2)\lambda + [\nu(\sigma h - g) + n^2\pi^2(n^2\pi^2\tilde{\omega}\mu + \tilde{\omega}\nu - \mu g)] = 0 \tag{32}$$

If $\tilde{\omega}$ and μ are non-zero, these eigenvalues tend to $-n^2\pi^2\tilde{\omega}$ and $-n^2\pi^2\mu$ (i.e. to $-\infty$) as $n \to \infty$, but if $\tilde{\omega} = 0$ one tends to g and the other to $+\infty$. We shall avoid the case of $\tilde{\omega} = 0$ for this reason.

We again ask how the uniform solution bifurcates and again have a diagram as in Fig. 4 for which the line OBC is

$$\mu = \left[\underset{n>0}{\mathrm{Min}^+} \frac{\sigma h - g + n^2\pi^2\tilde{\omega}}{(g - \tilde{\omega}n^2\pi^2)n^2\pi^2}\right]\nu = \left[\underset{n>0}{\mathrm{Min}^+} M(n)\right]\nu \tag{32}$$

Since $g < 1$, a necessary condition for instability by violation of (32) is $\tilde{\omega}\pi^2 < 1$. If μ is small enough, reduction of ν will take the solutions of the quadratic for $n = 0$ into the unstable half plane as a pair of complex conjugates. If μ is above the point B, a disturbance of wave number (n_0 say) that minimizes the right hand side of (31) begins to grown. This will be illustrated by some numerical examples, but we mention in passing that, using the usual type of analysis, a criterion to distinguish between sub- and supercriticality was obtained. It is too complicated a formula to permit a general conclusion, but it is suspected that as ν decreases the uniform steady-state gives up its stability to the non-uniform and that, whilst the uniform state is stable, stable non-uniform states do not exist when n_0 is odd but do exist when n_0 is even.

Three cases were calculated, with $\sigma = 2.5$, $\kappa = 0.144$ and $\tilde{\omega} = 0.01$, 0.02 and 0.05. When $\tilde{\omega} = 0.01$, $0 < M(2) < M(1)$, $M(n) < 0$, $n > 2$, so that the minimizing n_0 is 2 and we should expect a symmetric solution to arise first and perhaps give its stability to an unsymmetric one as ν decreases further since $M(1)$ is also positive. Fig. 14 is the result of transient calculations which converged to the stable non-uniform steady states. Though these are far from the transition it is the larger ν that gives the symmetric state $(n = 2)$. There is an analogy with the case of three compartments where, as ν decreases, the symmetric solutions (class III) are superceeded by the unsymmetric (class II).

For $\tilde{\omega} = 0.02$, $n_0 = 1$ and $M(n) < 0$ for $n > 1$. The non-uniform steady states depend only on the ratio μ/ν and three are shown in Fig. 15 that correspond to the rays in Fig. 16. The eigenvalues of the linearization about the non-uniform steady-state were calculated and it was found that stability was lost, by the real part of a pair of complex conjugate eigenvalues becoming positive, on the line BED. Since it was confirmed that the uniform periodic solution was stable for small enough ν, it seems likely that there is a region of complex behavior, KED, just as in the case $N = 2$. Calculations with $\tilde{\omega} = 0.05$ concur and the agreement between calculations for $N = 10$ and (D) give some confidence that this case of (C) gives a good indication of the dynamics of (D). Certainly the worst can be expected of (D) for which Pismen (13) has shown how chaos may arise.

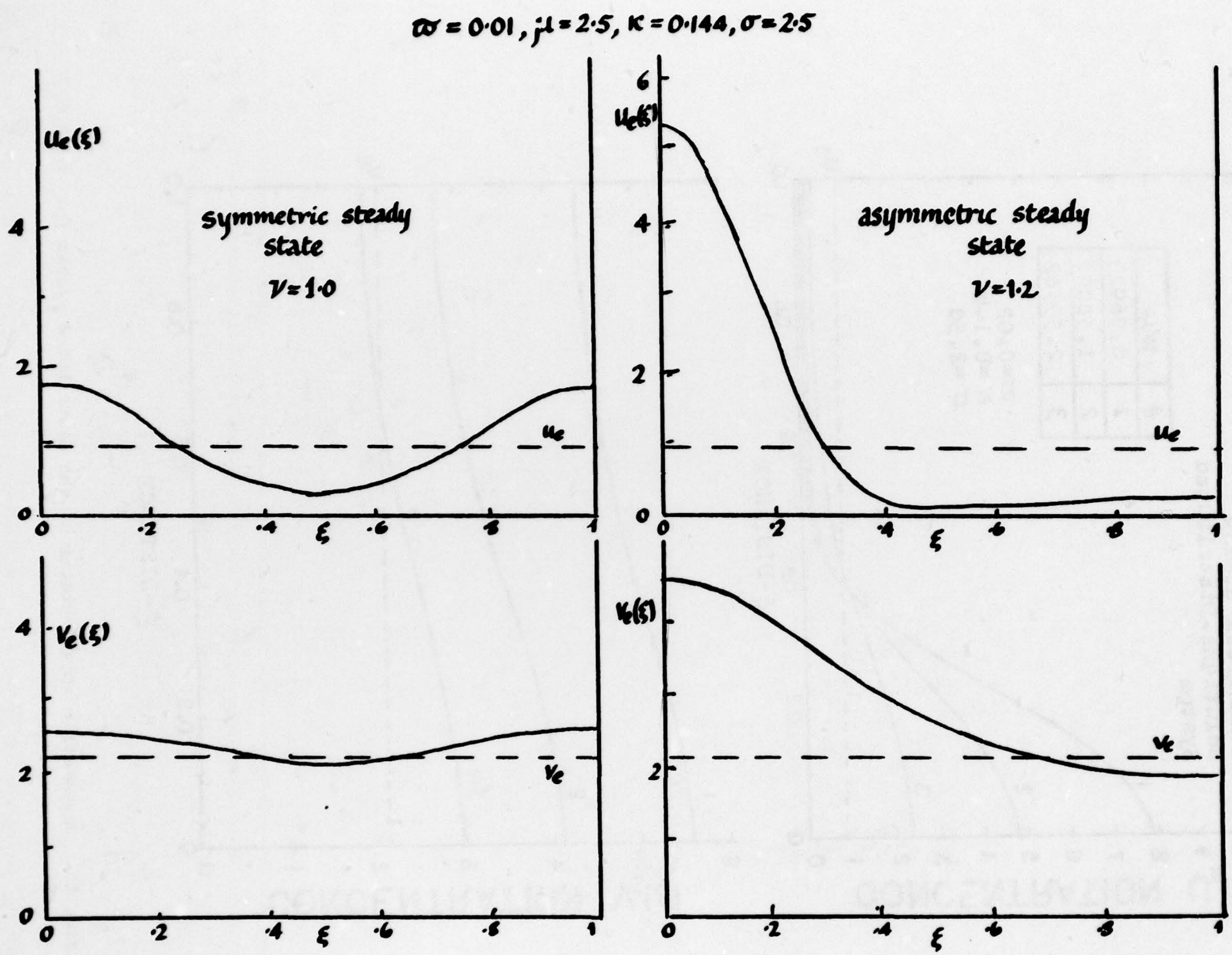


Figure 14 Symmetric and asymmetric steady-states for the distributed system with $\hat{\omega}$ equal to 0.01.

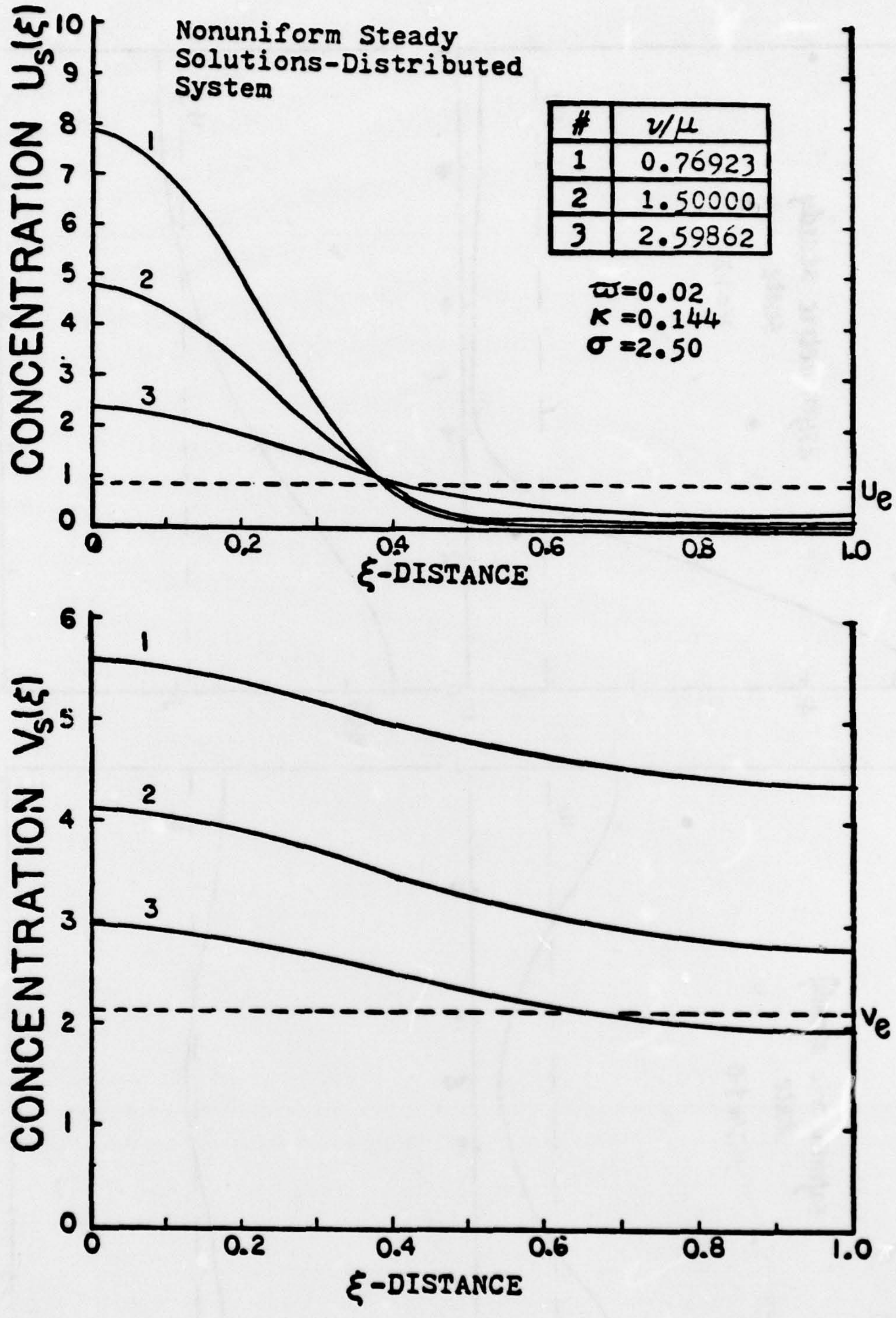


Figure 15 Unsymmetric steady-states of the distributed system for $\tilde{\omega} = 0.02$.

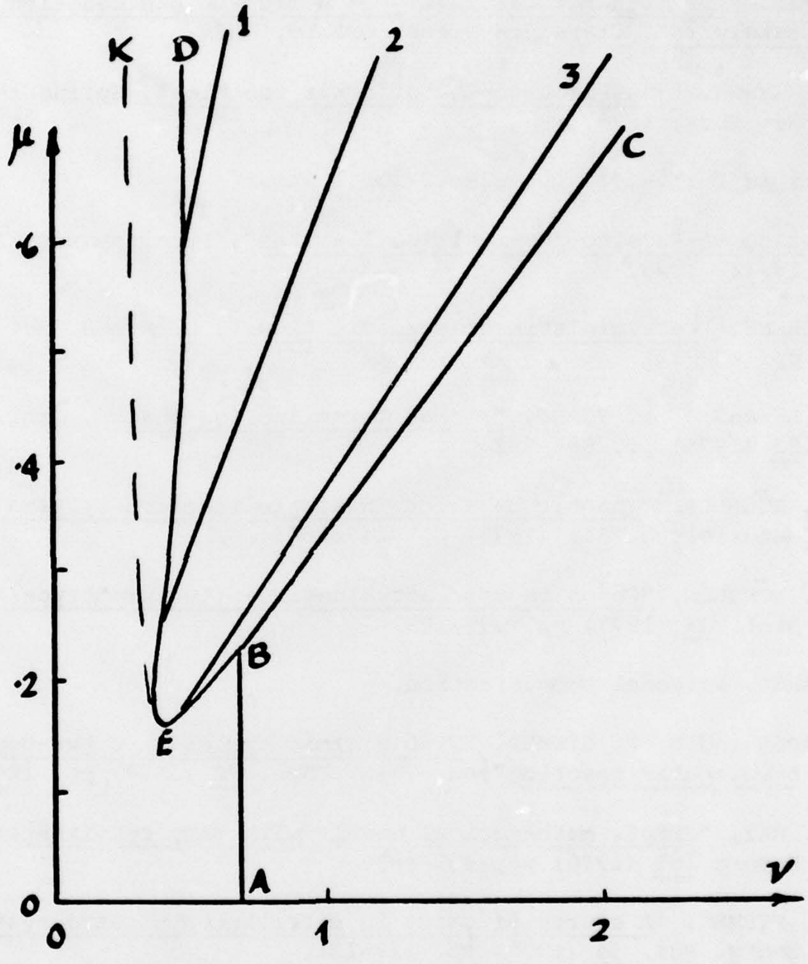


Figure 16 The presumed bifurcation diagram for the distributed system. The rays labeled 1, 2, and 3 are the loci of the solutions shown in Figure 15. The uniform solution is stable to the right of ABC, the uniform periodic solution to the left of KABA. The non-uniform steady solutions are stable within the arms of DEBC and a region of exotic behavior exists in KED.

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After some general observations on kinetic, compartmental and distributed systems a model suggested by Rössler is explored. Bifurcations to unsymmetric steady and uniform oscillatory states from the uniform steady states are demonstrated and the former are shown to break down into oscillations of increasing complexity.

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